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A STUDY ON SPECIAL FUNCTIONS WITH GENERATING RELATIONS FOR EXTON'S FUNCTION

Deepshikha Sharma
Research Scholar
Dr Sunita Pannu
Supervisor
Calorx Teacher's University

Abstract

Exton's functions, a class of hypergeometric functions, have garnered significant attention in mathematical physics and analysis due to their applications in various fields, including quantum mechanics, statistical mechanics, and number theory. This article will delve into a comprehensive study of special functions, particularly those related to Exton's functions, and explore the crucial role of generating relations in their characterization and applications. Exton's functions are generalizations of the classical hypergeometric functions, characterized by their specific parameters and functional forms. They often arise in the context of solving differential equations, integral representations, and other mathematical problems. Notable examples of Exton's functions include the Kampé de Fériet functions, the Lauricella functions, and the generalized hypergeometric functions. Generating relations are mathematical identities that express a given function in terms of an infinite series or integral involving simpler functions. These relations are indispensable in the study of special functions, providing a means to derive new properties, identities, and formulas. The development of generating relations for Exton's functions is a central topic in the field of special functions. These relations can be obtained through various techniques, such as the use of Laplace transforms, Mellin transforms, and differential equations. By establishing generating relations, we can express Exton's functions in terms of more familiar functions, such as the Gaussian hypergeometric function or the generalized hypergeometric function.

Keywords:

Exton; Functions; Hypergeometric; Laplace transforms; formulas

Introduction

Special functions are mathematical functions that arise in a wide range of scientific and engineering problems. They often exhibit specific properties and symmetries that make them particularly useful for solving differential equations, integrals, and other mathematical operations. Examples of well-known special functions include Bessel functions, Legendre polynomials, and Gamma functions.

We used the integral form of Exton's functions, which is a Laplace integral, in order to accomplish the task of determining the generating relations that exist between Exton's functions and hypergeometric functions. The hypergeometric functions of three variables, X1, X2,..., X20, below were defined by Exton in and integral representations were provided for them.

$$X_1(a,b;c,d;x,y,z)=\sum_{m,n,p=0}^{\infty}\frac{(a)_{2m+2n+p}(b)_p}{(c)_m(d)_{n+p}}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!},$$ ………………e.q.1.1

$$X_2(a,b;c_1,c_2,c_3;x,y,z)=\sum_{m,n,p=0}^{\infty}\frac{(a)_{2m+2n+p}(b)_p}{(c_1)_m(c_2)_n(c_3)_p}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!},$$ …e.q.1.2

$$X_3(a,b;c,d;x,y,z)=\sum_{m,n,p=0}^{\infty}\frac{(a)_{2m+n+p}(b)_{n+p}}{(c)_{m+n}(d)_p}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!},$$ …e.q.1.3

$$X_4(a,b;c_1,c_2,c_3;x,y,z)=\sum_{m,n,p=0}^{\infty}\frac{(a)_{2m+n+p}(b)_{n+p}}{(c_1)_m(c_2)_n(c_3)_p}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!},$$ …e.q.1.4

$$X_5(a,b_1,b_2;c;x,y,z)=\sum_{m,n,p=0}^{\infty}\frac{(a)_{2m+n+p}(b_1)_n(b_2)_p}{(c)_{m+n+p}}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!},$$ …e.q.1.5

$$X_6(a,b_1,b_2;c,d;x,y,z)=\sum_{m,n,p=0}^{\infty}\frac{(a)_{2m+n+p}(b_1)_n(b_2)_p}{(c)_{m+n}(d)_p}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!},$$ …e.q.1.6

$$X_7(a,b_1,b_2;c,d;x,y,z)=\sum_{m,n,p=0}^{\infty}\frac{(a)_{2m+n+p}(b_1)_n(b_2)_p}{(c)_m(d)_{n+p}}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!},$$,…e.q.1.7

$$X_8(a,b_1,b_2;c_1,c_2,c_3;x,y,z)=\sum_{m,n,p=0}^{\infty}\frac{(a)_{2m+n+p}(b_1)_n(b_2)_p}{(c_1)_m(c_2)_n(c_3)_p}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!},$$ …e.q.1.8

$$X_9(a,b;c;x,y,z)=\sum_{m,n,p=0}^{\infty}\frac{(a)_{2m+n}(b)_{n+2p}}{(c)_{m+n+p}}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!},$$ …e.q.1.9

$$X_{10}(a,b;c,d;x,y,z)=\sum_{m,n,p=0}^{\infty}\frac{(a)_{2m+n}(b)_{n+2p}}{(c)_{m+n}(d)_p}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!},$$ …e.q.1.10

$$X_{12}(a,b;c_1,c_2,c_3;x,y,z)=\sum_{m,n,p=0}^{\infty}\frac{(a)_{2m+n}(b)_{n+2p}}{(c_1)_m(c_2)_n(c_3)_p}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!},$$ ….….…e.q.1.11

$$X_{12}(a,b;c_1,c_2,c_3;x,y,z)=\sum_{m,n,p=0}^{\infty}\frac{(a)_{2m+n}(b)_{n+2p}}{(c_1)_m(c_2)_n(c_3)_p}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!},$$ …e.q.1.12

$$X_{13}(a,b,c;d;x,y,z)=\sum_{m,n,p=0}^{\infty}\frac{(a)_{2m+n}(b)_{n+p}(c)_p}{(d)_{m+n+p}}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!},$$ …e.q.1.13

$$X_{14}(a,b,c;d,d';x,y,z)=\sum_{m,n,p=0}^{\infty}\frac{(a)_{2m+n}(b)_{n+p}(c)_p}{(d)_{m+n}(d')_p}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!},$$ …e.q.1.14

$$X_{15}(a,b,c;d,d';x,y,z)=\sum_{m,n,p=0}^{\infty}\frac{(a)_{2m+n}(b)_{n+p}(c)_p}{(d)_m(d')_{n+p}}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!},$$ …e.q.1.15

$$X_{16}(a,b,c;d,d';x,y,z)=\sum_{m,n,p=0}^{\infty}\frac{(a)_{2m+n}(b)_{n+p}(c)_p}{(d)_{m+p}(d')_n}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!},$$ …e.q.1.16

$$X_{17}(a,b,c;d_1,d_2,d_3;x,y,z)=\sum_{m,n,p=0}^{\infty}\frac{(a)_{2m+n}(b)_{n+p}(c)_p}{(d_1)_m(d_2)_n(d_3)_p}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!},$$ …e.q.1.17

$$X_{18}(a,b,b',c;d;x,y,z)=\sum_{m,n,p=0}^{\infty}\frac{(a)_{2m+n}(b)_n(b')_p(c)_p}{(d)_{m+n+p}}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!},$$ …e.q.1.18

$$X_{19}(a,b,b',c;d,d';x,y,z)=\sum_{m,n,p=0}^{\infty}\frac{(a)_{2m+n}(b)_n(b')_p(c)_p}{(d)_m(d')_{n+p}}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!},$$ …e.q.1.19

$$X_{20}(a,b,b',c;d,d';x,y,z)=\sum_{m,n,p=0}^{\infty}\frac{(a)_{2m+n}(b)_n(b')_p(c)_p}{(d)_{m+n}(d')_n}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!},$$ …e.q.1.20

The integral form of these Exton's functions is shown in the next section, which makes use of the Laplace integral.

LAPLACE INTEGRAL REPRESENTATIONS

In this instance, we used the Laplace transform in order to express all of Exton's functions, X1, X2,..., X20, in terms of multiple integrals simultaneously. The following are some illustrations of the functions of Exton that are brought together.

$$X_1(a,b;c,d;x,y,z)=\frac{1}{\Gamma(a)\Gamma(b)}\int_0^{\infty}\int_0^{\infty}\exp(-s-t)s^{a-1}t^{b-1}$$
$$\cdot {}_0F_1(-;c;xs^2)\,{}_0F_1(-;d;ys^2+zst)dsdt,$$

$$X_2(a,b;c_1,c_2,c_3;x,y,z)=\frac{1}{\Gamma(a)}\int_0^{\infty}\exp(-s)s^{a-1}$$
$$\cdot {}_0F_1(-;c_1;xs^2)\,{}_0F_1(-;c_2;ys^2)\,{}_1F_1(b;c_3;zs)ds,$$ …e.q.1.21

$$X_3(a,b;c,d;x,y,z) = \frac{1}{\Gamma(a)\Gamma(b)} \int_0^\infty \int_0^\infty \exp(-s-t) s^{a-1} t^{b-1} \cdot {}_0F_1(-;c;xs^2+yst) {}_0F_1(-;d;zst) ds dt, \quad \text{...e.q.1.22}$$

$$X_4(a,b;c_1,c_2,c_3;x,y,z) = \frac{1}{\Gamma(a)} \int_0^\infty \exp(-s) s^{a-1} \cdot {}_0F_1(-;c_1;xs^2) \Psi_2(b;c_2,c_3;ys,zs) ds, \quad \text{...e.q.1.23}$$

$$X_5(a,b_1,b_2;c;x,y,z) = \frac{1}{\Gamma(a)\Gamma(b_1)\Gamma(b_2)} \int_0^\infty \int_0^\infty \int_0^\infty \exp(-s-t-u) \cdot s^{a-1} t^{b_1-1} u^{b_2-1} {}_0F_1(-;c;xs^2+yst+zsu) ds dt du, \quad \text{...e.q.1.24}$$

$$X_6(a,b_1,b_2;c,d;x,y,z) = \frac{1}{\Gamma(a)\Gamma(b_1)} \int_0^\infty \int_0^\infty \exp(-s-t) s^{a-1} t^{b_1-1} \cdot {}_0F_1(-;c;xs^2+yst) {}_1F_1(b_2;d;zs) ds dt, \quad \text{...e.q.1.25}$$

$$X_7(a,b_1,b_2;c,d;x,y,z) = \frac{1}{\Gamma(a)} \int_0^\infty \exp(-s) s^{a-1} \cdot {}_0F_1(-;c;xs^2) \Phi_2(b_1,b_2;d;ys,zs) ds, \quad \text{...e.q.1.26}$$

$$X_8(a,b_1,b_2;c_1,c_2,c_3;x,y,z) = \frac{1}{\Gamma(a)} \int_0^\infty \exp(-s) s^{a-1} \cdot {}_0F_1(-;c_1;xs^2) {}_1F_1(b_1;c_2;ys) {}_1F_1(b_2;c_3;zs) ds, \quad \text{...e.q.1.27}$$

$$X_9(a,b;c;x,y,z) = \frac{1}{\Gamma(a)\Gamma(b)} \int_0^\infty \int_0^\infty \exp(-s-t) s^{a-1} t^{b-1} \cdot {}_0F_1(-;c;xs^2+yst+zt^2) ds dt, \quad \text{...e.q.1.28}$$

$$X_{10}(a,b;c,d;x,y,z) = \frac{1}{\Gamma(a)\Gamma(b)} \int_0^\infty \int_0^\infty \exp(-s-t) \cdot s^{a-1} t^{b-1} {}_0F_1(-;c;xs^2+yst) {}_0F_1(-;d;zt^2) ds dt, \quad \text{...e.q.1.29}$$

$$X_{11}(a,b;c,d;x,y,z)=\frac{1}{\Gamma(a)\Gamma(b)}\int_0^\infty\int_0^\infty \exp(-s-t)s^{a-1}t^{b-1}$$
$$\cdot {}_0F_1(-;c;xs^2+zt^2){}_0F_1(-;d;yst)dsdt, \quad \text{...e.q.1.30}$$

$$X_{12}(a,b;c_1,c_2,c_3;x,y,z)=\frac{1}{\Gamma(a)\Gamma(b)}\int_0^\infty\int_0^\infty \exp(-s-t)s^{a-1}t^{b-1}$$
$$\cdot {}_0F_1(-;c_1;xs^2){}_0F_1(-;c_2;yst){}_oF_1(-;c_3;zs^2)dsdt, \quad \text{...e.q.1.31}$$

$$X_{13}(a,b,c;d;x,y,z)=\frac{1}{\Gamma(a)\Gamma(b)\Gamma(c)}\int_0^\infty\int_0^\infty\int_0^\infty \exp(-s-t-u)$$
$$\cdot s^{a-1}t^{b-1}u^{c-1}{}_0F_1(-;d;xs^2+yst+ztu)dsdtdu, \quad \text{...e.q.1.32}$$

$$X_{14}(a,b,c;d,d';x,y,z)=\frac{1}{\Gamma(a)\Gamma(b)}\int_0^\infty\int_0^\infty \exp(-s-t)s^{a-1}t^{b-1}$$
$$\cdot {}_0F_1(-;d;xs^2+yst){}_1F_1(c;d';zt)dsdt, \quad \text{...e.q.1.33}$$

$$X_{15}(a,b,c;d,d';x,y,z)=\frac{1}{\Gamma(a)\Gamma(c)}\int_0^\infty\int_0^\infty \exp(-s-t)s^{a-1}t^{c-1}$$
$$\cdot {}_0F_1(-;d;xs^2){}_1F_1(b;d';ys+zt)dsdt, \quad \text{...e.q.1.34}$$

$$X_{16}(a,b,c;d,d';x,y,z)=\frac{1}{\Gamma(a)\Gamma(b)\Gamma(c)}\int_0^\infty\int_0^\infty\int_0^\infty \exp(-s-t-u)$$
$$\cdot s^{a-1}t^{b-1}u^{c-1}{}_0F_1(-;d;xs^2+ztu){}_0F_1(-;d';yst)dsdtdu, \quad \text{...e.q.1.35}$$

$$X_{17}(a,b,c;d_1,d_2,d_3;x,y,z)=\frac{1}{\Gamma(a)\Gamma(c)}\int_0^\infty\int_0^\infty \exp(-s-t)$$
$$\cdot s^{a-1}t^{c-1}{}_0F_1(-;d_1;xs^2)\Psi_2(b;d_2,d_3;ys+zt)dsdt, \quad \text{...e.q.1.36}$$

$$X_{18}(a,b,b',c;d;x,y,z)=\frac{1}{\Gamma(a)\Gamma(b)\Gamma(b')\Gamma(c)}\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty \exp(-s)$$
$$\cdot \exp(-t-u-v)s^{a-1}t^{b-1}u^{b'-1}v^{c-1}{}_0F_1(-;d;xs^2+yst+zuv)dsdtdudv, \quad \text{...e.q.1.37}$$

$$X_{19}(a,b,b',c;d,d';x,y,z)=\frac{1}{\Gamma(a)\Gamma(c)}\int_0^\infty\int_0^\infty \exp(-s-t)$$
$$\cdot s^{a-1}t^{c-1}{}_0F_1(-;d;xs^2)\Phi_2(b,b';d';ys+zt)dsdt, \quad \text{...e.q.1.38}$$

$$X_{20}(a,b,b',c;d,d';x,y,z)=\frac{1}{\Gamma(a)\Gamma(b')\Gamma(c)}\int_0^\infty\int_0^\infty\int_0^\infty \exp(-s)$$
$$\exp(-t-u)s^{a-1}t^{b'-1}u^{c-1}{}_0F_1(-;d;xs^2+ztu){}_1F_1(b;d';ys)dsdtdu,$$ …e.q.1.39

These integral representations of Exton's functions were used in order to construct the new classes of generating relations, which are described in the next section in the form of relations.

NEW CLASSES OF GENERATING RELATIONS:

Here we find new classes of generating functions in the form of the following relations by using Exton's functions in the form of Laplace integrals;

$$\sum_{n=0}^{\infty}\frac{w^n}{n!}X_1(\alpha+n,\beta+n;\gamma,\delta;x^2,y,z)=(1+2x)^{-\alpha}\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{w}{1+2x}\right)^n$$ …e.q.1.40

$$\cdot X_6\left(\alpha+n,\beta+n,\gamma-\frac{1}{2};\delta,2\gamma-1;\frac{y}{(1+2x)^2},\frac{z}{1+2x},\frac{4x}{1+2x}\right)$$ …e.q.1.41

$$\sum_{n=0}^{\infty}\frac{w^n}{n!}X_1(\alpha+n,\beta+n;\gamma,\delta;x^2,y,z)$$
$$=\sum_{n,r=0}^{\infty}\frac{(\alpha+n)_{2r}w^nx^{2r}}{(\gamma)_r n!r!}H_3\left(\alpha+n+2r,\beta+n;\delta;y,z\right),$$ …e.q.1.42

$$\sum_{n=0}^{\infty}\frac{w^n}{n!}X_2(\alpha+n,\beta;\gamma,\delta,\lambda;x^2,y^2,z)$$
$$=(1-z)^{-\beta}\sum_{n=0}^{\infty}\frac{w^n}{n!}F_4\left(\frac{\alpha+n}{2},\frac{\alpha+n+1}{2};\gamma,\delta;4x^2,4y^2\right),$$ …e.q.1.43

$$\sum_{n=0}^{\infty} \frac{w^n}{n!} X_2(\alpha+n, \beta; \gamma, \delta, \lambda; x, y^2, z) = (1+2y)^{-\alpha} \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{w}{1+2y}\right)^n \cdot X_8 \left(\alpha+n, \beta, \delta-\frac{1}{2}; \gamma, \lambda, 2\delta-1; \frac{x}{(1+2y)^2}, \frac{z}{1+2y}, \frac{4y}{1+2y}\right), \quad \ldots\text{e.q.1.44}$$

$$= (1+2x+2y)^{-\alpha} \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{w}{1+2x+2y}\right)^n \cdot F_A^{(3)} \left(\alpha+n, \gamma-\tfrac{1}{2}, \delta-\tfrac{1}{2}, \beta; 2\gamma-1, 2\delta-1, \lambda; \tfrac{4x}{1+2x+2y}, \tfrac{4y}{1+2x+2y}, \tfrac{z}{1+2x+2y}\right) \quad \ldots\text{e.q.1.45}$$

$$\sum_{n=0}^{\infty} \frac{w^n}{n!} X_2(\alpha+n, \beta; \gamma, \delta, \lambda; x, y, z) = (1-z)^{-\alpha} \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{w}{1-z}\right)^n \cdot X_2 \left(\alpha+n, \lambda-\beta; \gamma, \delta, \lambda; \frac{x}{(1-z)^2}, \frac{y}{(1-z)^2}, \frac{z}{z-1}\right), \quad \ldots\text{e.q.1.46}$$

$$\sum_{n=0}^{\infty} \frac{w^n}{n!} X_3(\alpha+n, \beta+n; \gamma, \delta; x, y, zk) = (1+z)^{-\alpha}(1+k)^{-\beta} \cdot \sum_{n,r=0}^{\infty} \frac{(\beta+n)_r}{n!r!} \left(\frac{w}{(1+z)(1+k)}\right)^n \left(\frac{k}{1+k}\right)^r$$

$$\cdot X_6 \left(\alpha+n, \beta+n+r, \delta+r; \gamma, \delta; \frac{x}{(1+z)^2}, \frac{y}{(1+z)(1+k)}, \frac{z}{1+z}\right) \quad \ldots\text{e.q.1.47}$$

$$\sum_{n=0}^{\infty} \frac{w^n}{n!} X_3(\alpha+n, \beta+n; \gamma, \delta; x, y, zk) = (1+z)^{-\alpha}(1+k)^{-\beta} \cdot \sum_{n,r=0}^{\infty} \frac{(\beta+n)_r}{n!r!} \left(\frac{w}{(1+z)(1+k)}\right)^n \left(\frac{k}{1+k}\right)^r$$

$$\cdot X_6 \left(\alpha+n, \beta+n+r, \delta+r; \gamma, \delta; \frac{x}{(1+z)^2}, \frac{y}{(1+z)(1+k)}, \frac{z}{1+z}\right) \quad \ldots\text{e.q.1.48}$$

$$\sum_{n=0}^{\infty}\frac{w^n}{n!}X_4(\alpha+n,\beta;\gamma,\delta,\lambda;x^2,y,z)=(1+2x)^{-\alpha}\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{w}{1+2x}\right)^n$$
$$\cdot F_E\left(\alpha+n,\alpha+n,\alpha+n,\gamma-\tfrac{1}{2},\beta,\beta;2\gamma-1,\delta,\lambda;\tfrac{4x}{1+2x},\tfrac{y}{1+2x},\tfrac{z}{1+2x}\right), \quad \text{...e.q.1.49}$$

$$\sum_{n=0}^{\infty}\frac{w^n}{n!}X_4(\alpha+n,\beta;\gamma,\beta,\beta;x,y,z)=(1-y-z)^{-\alpha}\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{w}{1-y-z}\right)^n$$
$$\cdot F_4\left(\frac{\alpha+n}{2},\frac{\alpha+n+1}{2};\gamma,\beta;\frac{4x}{(1-y-z)^2},\frac{4yz}{(1-y-z)^2}\right), \quad \text{...e.q.1.50}$$

$$\sum_{n=0}^{\infty}\frac{w^n}{n!}X_4(\alpha+n,\beta;\gamma,\beta,\beta;x^2,y,y)=(1+2x)^{-\alpha}\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{w}{1+2x}\right)^n$$
$$\cdot F_2\left(\alpha+n,\beta-\frac{1}{2},\gamma-\frac{1}{2};2\beta-1,2\gamma-1;\frac{4y}{1+2x},\frac{4x}{1+2x}\right), \quad \text{...e.q.1.51}$$

$$\sum_{n=0}^{\infty}\frac{w^n}{n!}X_6(\alpha+n,\beta+n,\gamma;\epsilon,\delta;x,y,z)=(1-z)^{-\alpha}\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{w}{1-z}\right)^n$$
$$\cdot X_6\left(\alpha+n,\beta+n,\delta-\gamma;\epsilon,\delta;\frac{x}{(1-z)^2},\frac{y}{1-z},\frac{z}{z-1}\right), \quad \text{...e.q.1.52}$$

$$\sum_{n=0}^{\infty}\frac{w^n}{n!}X_6(\alpha+n,\beta+n,\gamma;\epsilon,\delta;x,y,z)=(1-z)^{-\alpha}\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{w}{1-z}\right)^n$$
$$\cdot H_3\left(\alpha+n,\beta+n;\epsilon;\tfrac{x}{(1-z)^2},\tfrac{y}{1-z}\right)\,{}_2F_1\left(\delta-\gamma,\alpha+n+2p+q;\delta;\tfrac{z}{z-1}\right), \quad \text{...e.q.1.53}$$

$$\sum_{n=0}^{\infty}\frac{w^n}{n!}X_6(\alpha+n,\beta+n,\gamma;\delta,\gamma;x,y,z)=(1-z)^{-\alpha}\sum_{n=0}^{\infty}\frac{1}{n!}$$
$$\cdot\left(\frac{w}{1-z}\right)^n H_3\left(\alpha+n,\beta+n;\delta;\frac{x}{(1-z)^2},\frac{y}{1-z}\right), \quad \text{...e.q.1.54}$$

$$\sum_{n=0}^{\infty}\frac{w^n}{n!}X_7(\alpha+n,\beta,\gamma;\delta,\lambda;x^2,y,z)=(1+2x)^{-\alpha}\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{w}{1+2x}\right)^n$$
$$\cdot F_G\left(\alpha+n,\alpha+n,\alpha+n,\delta-\tfrac{1}{2},\beta,\gamma;2\delta-1,\lambda,\lambda;\tfrac{4x}{1+2x},\tfrac{y}{1+2x},\tfrac{z}{1+2x}\right), \quad \text{...e.q.1.55}$$

$$\sum_{n=0}^{\infty}\frac{w^n}{n!}X_7(\alpha+n,\beta,\delta-\beta;\gamma,\delta;x,y,z)=(1-z)^{-\alpha}\sum_{n=0}^{\infty}\frac{1}{n!}\cdot\left(\frac{w}{1-z}\right)^n H_4\left(\alpha+n,\beta;\gamma,\delta;\frac{x}{(1-z)^2},\frac{y-z}{1-z}\right),$$

…e.q.1.56

$$\sum_{n=0}^{\infty}\frac{w^n}{n!}X_8(\alpha+n,\beta,\gamma;\delta,\lambda,\epsilon;x^2,y,z)=(1+2x)^{-\alpha}\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{w}{1+2x}\right)^n\cdot F_A^{(3)}\left(\alpha+n,\delta-\frac{1}{2},\beta,\gamma;2\delta-1,\lambda,\epsilon;\frac{4x}{1+2x},\frac{y}{1+2x},\frac{z}{1+2x}\right),$$

…e.q.1.57

$$\sum_{n=0}^{\infty}\frac{w^n}{n!}X_8(\alpha+n,\beta,\gamma;\delta,\lambda,\epsilon;x,y,z)=(1-y)^{-\alpha}\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{w}{1-y}\right)^n\cdot X_8\left(\alpha+n,\lambda-\beta,\gamma;\delta,\lambda,\epsilon;\frac{x}{(1-y)^2},\frac{y}{y-1},\frac{z}{1-y}\right),$$

…e.q.1.58

$$\sum_{n=0}^{\infty}\frac{w^n}{n!}X_8(\alpha+n,\beta,\gamma;\delta,\lambda,\epsilon;x,y,z)=(1-z)^{-\alpha}\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{w}{1-z}\right)^n\cdot X_8\left(\alpha+n,\beta,\epsilon-\gamma;\delta,\lambda,\epsilon;\frac{x}{(1-z)^2},\frac{y}{1-z},\frac{z}{z-1}\right),$$

…e.q.1.59

$$\sum_{n=0}^{\infty}\frac{w^n}{n!}X_8(\alpha+n,\beta,\gamma;\delta,\lambda,\epsilon;x,y,z)=(1-y-z)^{-\alpha}\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{w}{1-y-z}\right)^n\cdot X_8\left(\alpha+n,\lambda-\beta,\epsilon-\gamma;\delta,\lambda,\epsilon;\frac{x}{(1-y-z)^2},\frac{y}{y+z-1},\frac{z}{y+z-1}\right)$$

…e.q.1.60

$$\sum_{n=0}^{\infty}\frac{w^n}{n!}X_8(\alpha+n,\beta,\gamma;\delta,\beta,\lambda;x,y,z)=(1-y)^{-\alpha}\sum_{n=0}^{\infty}\frac{1}{n!}\cdot\left(\frac{w}{1-y}\right)^n H_4\left(\alpha+n,\gamma;\delta,\lambda;\frac{x}{(1-y)^2},\frac{z}{1-y}\right),$$

…e.q.1.61

$$\sum_{n=0}^{\infty} \frac{w^n}{n!} X_8(\alpha+n, \beta, \gamma; \delta, \lambda, \gamma; x, y, z) = (1-z)^{-\alpha} \sum_{n=0}^{\infty} \frac{1}{n!} \cdot \left(\frac{w}{1-z}\right)^n H_4\left(\alpha+n, \beta; \delta, \lambda; \frac{x}{(1-z)^2}, \frac{y}{1-z}\right),$$ …e.q.1.62

$$\sum_{n=0}^{\infty} \frac{w^n}{n!} X_8(\alpha+n, \beta, \gamma; \delta, \beta, \gamma; x, y, z) = (1-y-z)^{-\alpha} \sum_{n=0}^{\infty} \frac{1}{n!} \cdot \left(\frac{w}{1-y-z}\right)^n {}_2F_1\left(\frac{\alpha+n}{2}, \frac{\alpha+n+1}{2}; \delta; \frac{4x}{(1-y-z)^2}\right),$$ …e.q.1.63

$$\sum_{n=0}^{\infty} \frac{w^n}{n!} X_8(\alpha+n, \beta, \gamma; \delta, \beta, \gamma; x^2, y, z) = (1+2x-y-z)^{-\alpha} \sum_{n=0}^{\infty} \frac{1}{n!} \cdot \left(\frac{w}{1+2x-y-z}\right)^n {}_2F_1\left(\alpha+n, \delta-\frac{1}{2}; 2\delta-1; \frac{4x}{1+2x-y-z}\right),$$ …e.q.1.64

$$\sum_{n=0}^{\infty} \frac{w^n}{n!} X_8(\alpha+n, \beta, \gamma; \delta, \beta, \gamma; x^2, y, y) = (1+2x-2y)^{-\alpha} \sum_{n=0}^{\infty} \frac{1}{n!} \cdot \left(\frac{w}{1+2x-2y}\right)^n {}_2F_1\left(\alpha+n, \delta-\frac{1}{2}; 2\delta-1; \frac{4x}{1+2x-2y}\right),$$ …e.q.1.65

$$\sum_{n=0}^{\infty} \frac{w^n}{n!} X_9(\alpha+n, \beta+n; \gamma; x^2, 2xy, y^2) = (1+2x)^{-\alpha}(1+2y)^{-\beta} \sum_{n=0}^{\infty} \frac{1}{n!} \cdot \left(\frac{w}{(1+2x)(1+2y)}\right)^n F_1\left(\gamma-\frac{1}{2}, \alpha+n, \beta+n; 2\gamma-1; \frac{4x}{1+2x}, \frac{4y}{1+2y}\right)$$ …e.q.1.66

$$\sum_{n=0}^{\infty} \frac{w^n}{n!} X_9(\alpha+n, \beta+n; \gamma; x^2, 2x^2, x^2) = (1+2x)^{-(\alpha+\beta)} \sum_{n=0}^{\infty} \frac{1}{n!} \cdot \left(\frac{w}{(1+2x)^2}\right)^n {}_2F_1\left(\gamma-\frac{1}{2}, \alpha+\beta+2n; 2\gamma-1; \frac{4x}{1+2x}\right),$$ …e.q.1.67

$$\sum_{n=0}^{\infty} \frac{w^n}{n!} X_9(\alpha+n, 2\gamma-\alpha-n-1; \gamma; x^2, 2x^2, x^2)$$
$$= (1+2x)^{\frac{1}{2}-\gamma}(1-2x)^{\frac{1}{2}-\gamma} \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{w}{(1+2x)^2}\right)^n, \quad \text{…e.q.1.68}$$

$$\sum_{n=0}^{\infty} \frac{w^n}{n!} X_{10}(\alpha+n, \beta+n; \gamma, \delta; x, y, z^2) = (1+2z)^{-\beta} \sum_{n=0}^{\infty} \frac{1}{n!}$$
$$\cdot\left(\frac{w}{(1+2z)}\right)^n X_{14}\left(\alpha+n, \beta+n, \delta-\frac{1}{2}; \gamma, 2\delta-1; x, \frac{y}{1+2z}, \frac{4z}{1+2z}\right), \quad \text{…e.q.1.69}$$

Noting that, as a general rule, it is important to recognize that F_1, F_2, F_3, F_4 the functions that are carried out by the Appell. $F_A^{(3)}$ is the Lauricella function with three variables, 2F1 is the Gaussian hypergeometric function, and [62], pp.13(1), and pp.18(17) are sources that are pertinent to this discussion. $H_3,\ H_4$ Uses of Horn $F_E, F_G, F_K,\ F_M, F_N$ For further information, please refer to the three-variable functions that Saran presented In order to illustrate some of the discoveries that were discussed before, we will now go over the essential equations that are shown below. In order to get further information about these equations,

$$\sum_{m,n=0}^{\infty} f(m+n)\frac{x^m}{m!}\frac{y^n}{n!} = \sum_{N=0}^{\infty} f(N)\frac{(x+y)^N}{N!} \quad \text{…e.q.1.70}$$

$$\Psi_2(c; c, c; x, y) = \exp(x+y)\,{}_0F_1(-; c; xy)$$

$${}_1F_1(a; c; x) = \exp(x)\,{}_1F_1(c-a; c; -x)$$

$${}_0F_1(-; c; x^2) = \exp(-2x)\,{}_1F_1(c-\frac{1}{2}; 2c-1; 4x)$$

$$(\lambda)_n = \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)}, \quad \lambda \neq 0, -1, -2, ...$$

$$(\lambda)_{m+n} = (\lambda)_m(\lambda + m)_n$$

$$(\lambda)_{2n} = 2^{2n}(\frac{\lambda}{2})_n(\frac{\lambda+1}{2})_n, \quad n = 0, 1, 2, \ldots$$ …e.q.1.61

$$(n-k)! = \frac{(-1)^\kappa n!}{(-n)_k}, \quad o \leq k \leq n$$

$$L\{t^{2\sigma-1}{}_mF_n(\alpha_1, \ldots, \alpha_m; \beta_1, \ldots, \beta_n; \lambda^2 t^2)\} = \Gamma(2\sigma)p^{-2\sigma}$$

$${}_{m+2}F_n\left(\alpha_1, \ldots, \alpha_n, \tfrac{\sigma}{2}, \tfrac{\sigma+1}{2}; \beta_1, \ldots, \beta_n; 4\lambda^2 p^{-2}\right)$$

$$L\{t^{\gamma-1}{}_1F_1(\alpha, \gamma; \lambda t)\} = \Gamma(\gamma)p^{\alpha-\gamma}(p-\lambda)^{-\alpha},$$

$$Re(\gamma) > 0, \quad Re(p) > 0, \quad Re(\lambda) > 0.$$

$$L\{t^\upsilon\} = \Gamma(\upsilon+1)p^{-\upsilon-1}, \quad Re(\upsilon) > -1, Re(p) > 0.$$

$$L\{x^\mu {}_1F_1(a_1; b_1; \sigma x){}_1F_1(a; b; \omega x)\} = \Gamma(\mu+1)p^{-\mu-1}$$ …e.q.1.62

$$\times F_2\left(\mu+1, a, a_1; b, b_1; \tfrac{\omega}{p}, \tfrac{\sigma}{p}\right)$$

$$Re(\mu) > -1,\ Re(p-\sigma),\ Re(p-\omega),\ Re(p-\sigma-\omega) > 0.$$

Where L is the Laplace transform Φ_2, Ψ_2 2 correspond to the role of Humbert (for references, ${}_1F_1$ according to Kummer, the function (or functions) that are confluent hypergeometric. also, ${}_mF_n$ The application of the generalized hypergeometric function (for further information,

Proof of result

By using the formula and the symbol I, which stands for the left-hand side of the equation we are able to get

$$I=\sum_{n=0}^{\infty}\frac{w^n}{n!\Gamma(\alpha+n)\Gamma(\beta+n)}\int_0^{\infty}\int_0^{\infty}\exp(-s-t)s^{\alpha+n-1}t^{\beta+n-1}$$

$$\cdot{}_0F_1(-;\gamma;x^2s^2){}_0F_1(-;\delta;ys^2+zst)dsdt.$$ …e.q.1.63

The functions ${}_1F_1$ and ${}_0F_1$ It is possible to represent the integrand as a series by applying the formula which is included in the analysis. After that, we are able to get it by rearranging the order of the integrals and summations, which is the appropriate method in this particular setting.

$$I=\sum_{n,p,q,r=0}^{\infty}\frac{(\gamma-\frac{1}{2})_p(4x)^pw^ny^qz^r}{(2\gamma-1)_p(\delta)_{q+r}n!p!q!r!\Gamma(\alpha+n)\Gamma(\beta+n)}\int_0^{\infty}\int_0^{\infty}\exp(-t)$$
$$\cdot\exp(-s(1+2x))s^{\alpha+n+p+2q+r-1}t^{\beta+n+r-1}dsdt.$$ …e.q.1.64

Additionally, we apply the equations in the preceding equation, and then we reduce it by manipulating series, which brings the proof of result to a comprehensive conclusion.

Remark By taking into account the suitable integral representation and Laplace transform while the proof is being carried out, the proof of the remaining findings is carried out in a manner that is comparable.

Conclusion

The study of special functions, particularly those related to Exton's functions, is a rich and active area of research with a wide range of applications. Generating relations play a fundamental role in the characterization and understanding of these functions, providing powerful tools for deriving new properties, identities, and formulas. By exploring the connections between Exton's functions and generating relations, we can deepen our knowledge of these mathematical objects and their applications in diverse fields.

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